Lessons from Mathematics for Decision making

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*Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth.-**Arthur Conan Doyle***

Decision making is the difficult part; we can recall the story of the ***Jinn*** (Genie), who got mad when his master asked him to segregate the bad potatoes from the good ones. He pleaded for mercy and asked him for anything else but decision making. Decision making is based on a complex search for information, uncertainty, conflicting requirements and individual's personal preference. There are several strategies in Mathematics which help in problem solving.

1. Pattern search
2. Drawing a figure
3. Formulate an equivalent problem
4. Modify the problem
5. Choose effective notation
6. Divide the problem
7. Reverse strategy or work backwards
8. Contradictory method
9. Extreme case consideration
10. Generalize

We will not go into details of each technique but through some famous puzzles understand the process of arriving at a solution which helps in decision making. Training one's mind to think mathematically can greatly enhance the decision making skills. Thinking mathematically does not entail knowing the great formulae and methods of mathematics, but just being logical and applying common sense. A simple example from algebra can illustrate the power of reverse strategy, a problem solving strategy in mathematics. There are two ways, "peasant's way" and the "poet's way" to solve a problem. One is the normal way, other the ***"beautiful"*** way.

The sum of two numbers is 2, the product of the same two numbers is 3. What is the sum of the reciprocals of the two numbers?

The standard way would be, let, x be the first number, and y, the second number

The sum of the two numbers is 2.

So, $x + y = 2$, and product of the two numbers is 3: so, $xy = 3$

The sum of the reciprocals of the two numbers is $(1/x) + (1/y)$

The peasant's way (natural, normal way) is substituting the value of y in xy=3, which gives,

$x^2-2x+3=0$, and then using the quadratic formula etc.…

But the poet's way (the beautiful way) is thinking reverse,

$1/x + 1/y = (x+y)/xy$, now it's so simple, answer is there, 2/3.

This is known as reverse strategy, start from the end.

Rearranging the problem is another interesting method. A good example follows.

The problem is to find the sum of numbers sequentially from 1-99:

1+2+3+…..+97+98+99

This will surely take some time even when it's not so difficult.

What if we arrange the terms like this,

(1+99) + (2+98) + (3+97) +....+ (48+52) + (49+51) + 50

Even without the calculator, it's easy now. There are 49 terms adding upto 100 and one 50, so the answer is 4900+50=4950. So simple and straight.

LACK of Information- Example of Radius of Circle

Sometimes just a glance on the problem makes us believe that there is lack of information. Moreover there can never be a state of perfect or complete information. It may be possible to solve a problem by assimilating knowledge. A good example is in the Figure I. Find the radius of circle if the rectangle at the corner measures 6 cm x 12 cm.

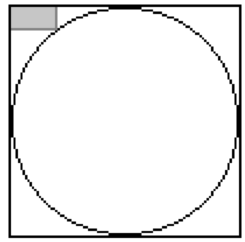

Figure I

This seems like an incomplete problem at the first instance, but a closer scrutiny and the ensuing solution will show how simple the problem was,

The Pythagoras' Theorem is one of the most famous theorems in geometry and a simple use of the theorem will solve this.

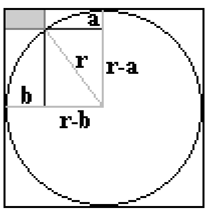

Figure II

Construct a right angled triangle as shown in Fig II

We have, $(r - a)^2 + (r - b)^2 = r^2$

Now, $a = 6$ cm and $b = 12$ cm. So,

$(r - 6)^2 + (r - 12)^2 = r^2$ or

$r^2 - 2(6)r + 6^2 + r^2 - 2(12)r + 12^2 = r^2$. This simplifies to

$r^2 - 36r + 180 = 0$. On factorizing, we get

$(r - 30)(r - 6) = 0$.

Thus, the radius of the circle is 30 cm as it cannot obviously be 6.

What you believe is the best is not the best

When we solve a problem and shout "Eureka", be careful it may not be the best solution. It is important to reconsider the decision taken. This Bridge Crossing at Night problem will illustrate the dictum.

A group of four people, who have one flashlight, need to cross a bridge at night. A maximum of two people can cross the bridge at one time, and any party that crosses (either one or two people) must have the flashlight with them. The flashlight is a must and without that it is not possible to cross the bridge. Person A takes 1 minute to cross the bridge, person B takes 2 minutes, person C takes 5 minutes, and person D takes 8 minutes. A pair must walk together at the rate of the slower person's pace. What is the minimum time they take to accomplish the task?

There is an interesting part of this problem solving, if no time is given the first answer that one gets is believed to be the best. But there is still a faster way.

Let's see, this is the natural way to solve. The fastest movers are moving together and again moving with the slower ones.

Elapsed Time	Starting Side	Movement	Ending Side
0 minutes	A B C D		
2 minutes	C D	A and B cross forward, taking 2 minutes	A B
3 minutes	A C D	A returns, taking 1 minute	B
8 minutes	D	A and C cross forward, taking 5 minutes	A B C
9 minutes	A D	A returns, taking 1 minute	B C
17 minutes		A and D cross forward, taking 8 minutes	A B C D

But, if the question mentions that it can be accomplished in 15 minutes, then you will be forced to rethink. A slight change in strategy is required, the two slowest people crossing individually wastes time, thus they must cross together.

Elapsed Time	Starting Side	Movement	Ending Side
0 minutes	A B C D		
2 minutes	C D	A and B cross forward, taking 2 minutes	A B
3 minutes	A C D	A returns, taking 1 minute	B
11 minutes	A	C and D cross forward, taking 8 minutes	B C D
13 minutes	A B	B returns, taking 2 minutes	C D
15 minutes		A and B cross forward, taking 2 minutes	A B C D

So you save 2 minutes and now you know what you initially thought was not optimal. Hence even without boundary conditions, a rethink on the strategy adopted must be done.

Another example of checking the optimality

Given an 8 litre jug full of water and two empty jugs of 5- and 3-litre capacity, get exactly 4 litre of water in one of the jugs by completely filling up and/or emptying jugs into others.

It’s easy but the key is in finding the optimal solution with least number of moves.

8 litre	*5 litre*	*3 litre*	*Remarks*
8	0	0	
3	5	0	
3	2	2	
6	2	0	
6	0	2	
1	5	2	3 litre jar has space for 1 litre
1	4	3	
4	4	0	

The simplicity of the problem, keep it simple.

Knowledge of big formulae and strategies can in fact sometime slow down the decision making, it's better to start and assume simplicity in the problem. This problem of the flight of the bird will illustrate how solving can be simplified.

A train leaves City X for City Y which are 350 km apart, at 15 kmph. At the very same time, a train leaves City Y for City X at 20 kmph on the same track. At the same moment, a bird leaves the City X train station and flies towards the City Y train station at 25 kmph. When the bird reaches the train from City Y, it immediately reverses direction. It then continues to fly at the same speed towards the train from City X, when it reverses its direction again, and so forth. The bird continues to do this until the trains collide. How far would the bird have traveled in the meantime?

Knowledge of mathematics can slow down the time taken for solving, as it is natural to start summing the infinite series. But if the mind keeps it simple, it's really that simple,

Time elapsed before trains collide can be estimated by dividing total distance 350km by the relative speed of trains i.e., 15+20=35 kmph. Thus it is 10 hours.

So before colliding the bird would have flown for 10 hours i.e. 10×25=250 km. STRAIGHT and SIMPLE.

The story goes that John von Neumann, a pioneer of computer science was asked a similar problem and he solved it using infinite series.

The Census Taker - unsolvable, lacks coherent data

Let's consider this strange problem, where the statements do not seem to be related. A census taker approaches a woman at her house and asks about her children. She says, "I have three children and the product of their ages is thirty six. The sum of their ages is the number on this gate." The census taker does some calculation and claims not to have enough information. The woman then tells him, "My eldest child is good at football." Bewildering, how on the earth can someone know the ages.

But he knows the age of children, it seems improbable.

Let's look at the solution,

The product of the ages is 36, so there are only a few possible triples of ages. Here is a table of all the possibilities, with the sums of the ages below each triple.

(1,1,36)	(1,2,18)	(1,3,12)	(1,4,9)	(1,6,6)	(2,2,9)	(2,3,6)	(3,3,4)
38	21	16	14	13	13	11	10

The mother's second statement would have been enough to guess the age for most of combinations as they have unique solutions. But it is evident that the ages are either (1,6,6) or (2,2,9), for in all other cases, knowledge of the sum would unambiguously reveal the ages. The final clue that there *is* an eldest child, eliminates the option (1,6,6). The children are thus 2, 2 and 9 years old.

Another example of insufficient data and seeming impossibility.

A man walked five hours, first along a level road, then up a hill, then he turned round and walked back to his starting point along the same route. He walks 4 km per hour on the level, 3 uphill, and 6 downhill. Find the distance walked.

Is this a reasonable problem? Is the data sufficient to determine the unknown?

The data seem to be insufficient: some information about the extent of the non level part of the route seems to be lacking. If we knew how much time the man spent walking uphill, or downhill, there would be no difficulty. Yet without such information the problem appears indeterminate.

Still, let us try.

Let

x stand for the total distance walked,

y for the length of the uphill walk.

The walk had four different phases: level, uphill, downhill, level.

Now we can easily express the total time spent in walking,

$$\frac{x/2 - y}{4} + \frac{y}{3} + \frac{y}{6} + \frac{x/2 - y}{4} = 5$$

Just one equation between two unknowns -it is insufficient. Yet, when we collect the terms, the coefficient of ***y*** turns out to be 0, and there remains

$$\frac{x}{4} = 5 \; or \; x = 20$$

And so the data are sufficient to determine ***x***, the only unknown required. This obviously is a special case, not true in all conditions. So, the lesson is data insufficiency cannot be assumed unless tried really hard. Concluding that there is no solution has to be a conscious decision, but only after trying.

Sometimes it is not as simple as it seems, be careful,

In decision making, the obvious may not be right, holistic view of the problem is must. The example of bicyclist's average speed will illustrate this point.

A bicyclist goes up a hill at 30 kmph and down the same hill at 90 kmph. What is the cyclist's average speed for the trip?

At first, you might think that the answer is the simple average of 30 kmph and 90 kmph, i.e., 60 kmph. But this isn't correct since the cyclist spends less time at the faster rate. A quick way to find the average speed is to assume that the answer is independent of the length of the hill. If that's true, then we can set the length of the hill to a convenient value, say, 90 km. Then the trip takes 3 hours up the hill and 1 hour down. So the average speed is 180 km /4 hr = 45 kmph.

LOOSEN up your thoughts,

The artificial boundary that we create around ourselves has to be broken. Lateral thought is must for decision making. Connecting all nine points in Figure III with an unbroken path of four straight lines is impossible unless you liberate yourself from the artificial boundary of the nine points. Once you decide to draw lines that extend past this boundary, it is pretty easy. Let the first line join three points, and make sure that each new line connects two more points.

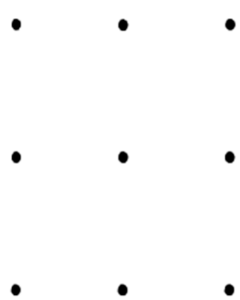

Figure III

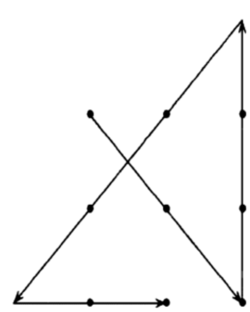

Figure IV

Another example of breaking the boundaries, A boy wants to take a 1.5 metre long sword in a train, but the conductor won't allow it as carry-on luggage. And the baggage person won't take any item whose greatest dimension exceeds 1 meter. What should the boy do?

This is unsolvable if we limit ourselves to two-dimensional space. Once liberated from 2D space, we get a nice solution: The sword fits into a 1 x 1 x 1-metre *box,* with a long diagonal of $\sqrt{1^2+1^2+1^2} = \sqrt{3} > 1.69\ m$. All that the boy has to do is get this box made.

IF IT SEEMS IMPOSSIBLE, DON'T GIVE UP – Rearrange

Consider the following diagram. Can you connect each small box on the top with its same-letter mate on the bottom with paths that do not cross one another, nor leave the boundaries of the large box?

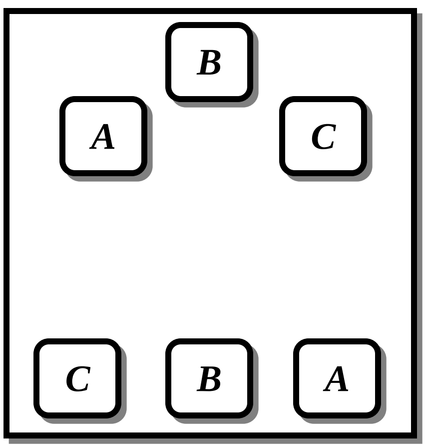

Figure V

It does seem impossible,

Let's try to make is simple

Get the upper C closer to the one below as in Fig VI. Join them.

Now join A and B, which is fairly simple

Topologically push C back to its original place and you get Fig VII. SOLVED

That happened for a mathematical reason: the problem was a "topological" one. This trick is to mutate the diagram into a "topologically equivalent" one.

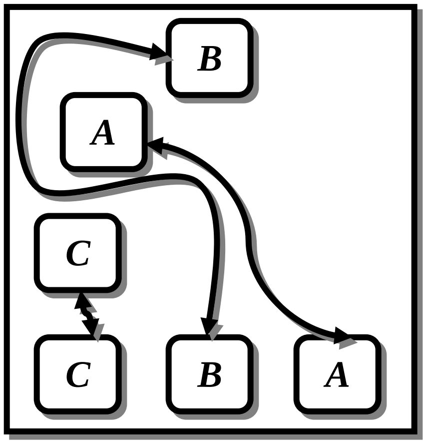

Figure VI

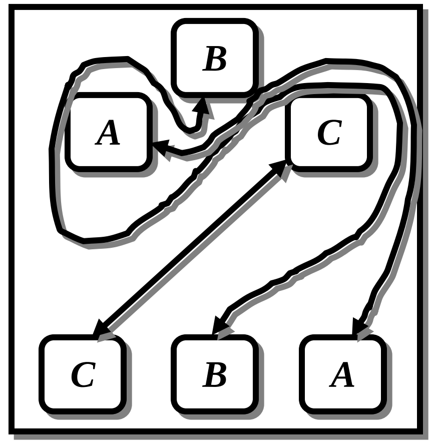

Figure VII

It is not a strategy, but rather a *tool*, in mathematics. Thus modifying the problem or rearranging the problem can make the impossible, possible. Another example illustrating the point.

A square is inscribed in a circle that is inscribed in a square. Find the ratio of the areas of the two squares

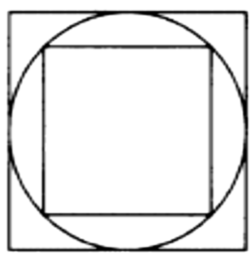

Figure VIII

This can be solved algebraically, but if the orientation of the inner square is changed or rotated by 90˚, just sense how simple it has become,

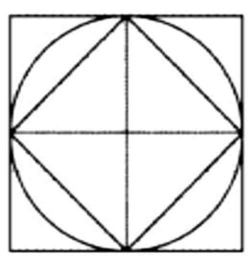

Figure IX

Obviously without any calculation, Inner Square is half of outer square. Rearranging the problem saved so much time and effort. The obvious was thus visible without an iota of calculation.

Sometimes elevating a problem leads to a solution

Here is a paradox, elevating the problem, leads to solution. All this while, simplifying, modifying, rearranging etc. has been discussed and now this anti-thesis. A good decision maker is one who knows all techniques and develops the insight on their utility. Let's try this problem.

Which number is greater?

$$\sqrt{6} + \sqrt{10} \; or \; \sqrt{5} + \sqrt{12}$$

A straight way is to calculate the square roots and compare, but that's not a beautiful way to solve.

Instead if the two terms are squared, then many roots will be eliminated and it will be straight to compare.

$$(\sqrt{6} + \sqrt{10})^2 = 6 + 2\sqrt{60} + 10 = 16 + 2\sqrt{60}$$

and

$$(\sqrt{5} + \sqrt{12})^2 = 5 + 2\sqrt{60} + 12 = 17 + 2\sqrt{60}$$

It's so obvious now that the second term is larger.

Lesson: Sometimes upgrading the method helps in solving a tough or a complex problem.

Every problem needs a solution, even if the solution leads to "no" solution. The honest attempt is important. Decision making is a science and a mathematically trained mind can help in reducing time for decisions, finding optimal solutions. Heuristics are important in problem solving, but reduction in discretion is the aim of the whole exercise.

Suggested Reading

Cofman, Judita (1990). What to Solve?: Problems and Suggestions for Young Mathematicians. A volume in the series, Oxford science publications, Oxford University Press, Oxford

Engel, Arthur (1998). Problem-solving Strategies, A volume in the series Problem Books in Mathematics, Springer-Verlag, New York

Krantz, Steven G. (1997). Techniques of Problem Solving. American Mathematical Society

Larson, Loren C. (1983). Problem Solving Through Problems, 3rd edition, Volume 5 of the series Problem Books in Mathematics, Springer-Verlag, New York

Polya, G. (1945). How to Solve It. Princeton, NJ: Princeton University Press.

Polya, G. (1954). Mathematics and plausible reasoning (Volume 1, Induction and analogy in mathematics; Volume 2, Patterns of plausible inference). Princeton: Princeton University Press.

Polya, G. (1962). Mathematical discovery: On understanding, learning and teaching problem solving: Volume I. New York: John Wiley and Sons, Inc.

Schoenfeld, A. H. (1985). Mathematical problem solving. New York: Academic Press.

Zeitz, Paul (1999). The Art and Craft of Problem Solving. John Wiley & Sons, New York

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